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A NOTE ON COOPERATIVE LINEAR QUADRATIC CONTROL

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A note on cooperative linear quadratic control.

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Abstract In this note we consider the cooperative linear quadratic control problem. That is, the problem where a number of players, all facing a (different) linear quadratic control problem, decide to cooperate in order to optimize their performance. It is well-known, in case the performance criteria are positive definite, how one can determine the set of Pareto efficient equilibria for these games. In this note we generalize this result for indefinite criteria.

Keywords: *LQ theory; Riccati equations; Cooperative differential games; Pareto frontier.*

Jel-codes: C61, C71, C73.

1 Introduction

In this note we assume that the performance criterion player $i = 1, 2$ likes to minimize is:

$$J_i(u_1, u_2) := \int_0^T [x^T(t), u_1^T(t), u_2^T(t)] M_i \begin{bmatrix} x(t) \\ u_1(t) \\ u_2(t) \end{bmatrix} dt, \quad (1)$$

where $M_i = \begin{bmatrix} Q_i & V_i & W_i \\ V_i^T & R_{1i} & N_i \\ W_i^T & N_i^T & R_{2i} \end{bmatrix}$ is symmetric, $\begin{bmatrix} R_{1i} & N_i \\ N_i^T & R_{2i} \end{bmatrix} > 0$, $i = 1, 2$, and $x(t)$ is the solution of the linear differential equation

$$\dot{x}(t) = Ax(t) + B_1 u_1(t) + B_2 u_2(t) + c(t), \quad x(0) = x_0. \quad (2)$$

The variable $c(\cdot) \in L_2$ is some given trajectory. Notice that we make no definiteness assumptions w.r.t. matrix Q_i . The planning horizon, T , may be either finite or infinite.

The objectives of the players are possibly conflicting. That is, a set of policies u_1 which is optimal for player one, may have rather negative effects on the evolution of the state variable x from another player's point of view.

Before one can analyze the outcome of such a decision process, a number of points have to be made more clear. We assume that players can communicate and can enter into binding agreements. Furthermore it is assumed that they cooperate in order to achieve their objectives. However, no

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side-payments take place. Moreover, it is assumed that every player has all information on the state dynamics and cost functions of his opponents and all players are able to implement their decisions. Concerning the strategies used by the players we assume that there are no restrictions. That is, every $u_i(\cdot)$ may be chosen arbitrarily from a set \mathcal{U} (which depends on the problem setting and will be specified later) in order to have a well-posed problem.

By cooperation, in general, the cost a specific player incurs is not uniquely determined anymore. If all players decide, e.g., to use their control variables to reduce the cost of player 1 as much as possible, a different minimum is attained for player 1 than in case all players agree to help collectively a different player in minimizing his cost. So, depending on how the players choose to "divide" their control efforts, a player incurs different "minima". So, in general, each player is confronted with a whole set of possible outcomes from which somehow one outcome (which in general does not coincide with a player's overall lowest cost) is cooperatively selected. Now, if there are two strategies γ_1 and γ_2 such that every player has a lower cost if strategy γ_1 is played, then it seems reasonable to assume that all players will prefer this strategy. We say that the solution induced by strategy γ_1 *dominates* the solution induced by the strategy γ_2 . So, dominance means that the outcome is better for all players. Proceeding in this line of thinking, it seems reasonable to consider only those cooperative outcomes which have the property that if a different strategy than the one corresponding with this cooperative outcome is chosen, then at least one of the players has higher costs. Or, stated differently, to consider only solutions that are such that they can not be improved upon by all players simultaneously. This motivates the concept of Pareto efficiency.

Definition 1.1 Let \mathcal{U} denote the set of admissible strategies. A set of strategies $\hat{\gamma}$ is called *Pareto efficient* if the set of inequalities

$$J_i(\gamma) \leq J_i(\hat{\gamma}), \quad i = 1, \dots, N,$$

where at least one of the inequalities is strict, does not allow for any solution $\gamma \in \mathcal{U}$. The corresponding point $(J_1(\hat{\gamma}), \dots, J_N(\hat{\gamma})) \in \mathbb{R}^N$ is called a *Pareto solution*. The set of all Pareto solutions is called the *Pareto frontier*. \square

A Pareto solution is therefore never dominated, and for that reason called an *undominated* solution. Usually there is more than one Pareto solution, because dominance is a property which generally does not provide a total ordering.

In case $Q_i \geq 0$, $R_{ii} > 0$, $i = 1, 2$, and all other matrices in our cost functions (1) are zero there is a simple characterization for all Pareto solutions (see e.g. [1]). In this note we will generalize this result.

2 Problem Solution

In the subsequent analysis the following set of parameters, \mathcal{A} , plays a crucial role.

$$\mathcal{A} := \{\alpha = (\alpha_1, \dots, \alpha_N) \mid \alpha_i \geq 0 \text{ and } \sum_{i=1}^N \alpha_i = 1\}.$$

The next two lemmas provide a characterization of Pareto efficient solutions. The first lemma shows how Pareto efficient solutions can be identified (see also [5] and [13]). This lemma holds without using

any convexity conditions on the J'_i 's nor any convexity assumptions regarding the strategy space. Its converse, Lemma 2.2 was proved by Fan et al in [2]. This lemma states that, under some convexity assumptions on the cost functions, all Pareto efficient strategies can be obtained by considering the minimization problem (3). A proof of both results can be found in, e.g., [1] (see also [9]).

Lemma 2.1 *Let $\alpha_i \in (0, 1)$, with $\sum_{i=1}^N \alpha_i = 1$. Assume $\hat{\gamma} \in \mathcal{U}$ is such that*

$$\hat{\gamma} \in \arg \min_{\gamma \in \mathcal{U}} \left\{ \sum_{i=1}^N \alpha_i J_i(\gamma) \right\}. \quad (3)$$

Then $\hat{\gamma}$ is Pareto efficient. □

Lemma 2.2 *Assume that the strategy space \mathcal{U} is convex¹. Moreover, assume that the payoffs J_i are convex, $i = 1, \dots, N$. Then, if $\hat{\gamma}$ is Pareto efficient, there exist $\alpha \in \mathcal{A}$, such that for all $\gamma \in \mathcal{U}$*

$$\sum_{i=1}^N \alpha_i J_i(\gamma) \geq \sum_{i=1}^N \alpha_i J_i(\hat{\gamma}). \quad \square$$

Note, that whenever J_i are convex also $\sum_{i=1}^N \alpha_i J_i(\gamma)$ is convex for an arbitrary $\alpha \in \mathcal{A}$.

Next we consider some sets of control functions that are relevant in our problem setting. It can be easily shown that each of these (nonempty²) sets is convex.

Lemma 2.3 *Let \mathcal{U} be given by either:*

- 1) $L_2[0, T] := \{(u_1, u_2) \mid u_i(\cdot) \text{ are square integrable functions on } [0, T]\}$.
- 2) $\{u \in L_2[0, T] \mid x(0) = x(T) = 0 \text{ in } \dot{x}(t) = Ax(t) + Bu(t)\}$.
- 3) $L_{2,e}^+ := \{(u_1, u_2) \mid u_i(\cdot) \in L_{2,loc} \text{ and } \lim_{T \rightarrow \infty} J_i(x_0, u) \text{ exists in } \mathbb{R} \cup \{-\infty, \infty\}, \forall x_0\}$, where $L_{2,loc}$ is the set of locally square-integrable functions, i.e.,

$$L_{2,loc} = \{u \mid \forall T > 0, \int_0^T u^T(s)u(s)ds < \infty\}.$$

- 4) $L_{2,e,s}^+ := \{(u_1, u_2) \mid (u_1, u_2) \in L_{2,e}^+ \text{ and } \lim_{t \rightarrow \infty} x(t) = 0 \text{ in } (2)\}$.

Then each of these set of control functions is convex. □

Next we consider the question under which conditions the cost functions J_i in (1) are convex. To that end we first derive some preliminary results.

Lemma 2.4 *Assume \mathcal{U} is convex. Consider the linear quadratic cost function*

$$J(s, T, u, x_0) = \int_s^T \{[x^T(t) \ u^T(t)]M \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + 2[p_1^T(t) \ p_2^T(t)] \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + c_2(t)\}dt \quad (4)$$

subject to the state dynamics

$$\dot{x}(t) = Ax(t) + Bu(t) + c_1(t), \quad x(s) = x_0, \quad (5)$$

¹Note that if \mathcal{U}_i is convex also the Cartesian product $\mathcal{U} := \mathcal{U}_1 \times \mathcal{U}_2$ is a convex set.

²assuming for case 2) and 3) that $(A, [B_1 \ B_2])$ is stabilizable

where u , p_i and c_i are such that (4) and (5) have a solution; $M = \begin{bmatrix} Q & V \\ V^T & R \end{bmatrix}$.

Let $x_0 \in \mathbb{R}^N$. Then, $J(0, T, u, x_0)$ is convex as a function of u if and only if $\bar{J}(0, T, v, 0) \geq 0$ for all v , where

$$\bar{J}(s, T, v, 0) = \int_s^T [z^T(t) \ v^T(t)] M \begin{bmatrix} z(t) \\ v(t) \end{bmatrix} dt \text{ with } \dot{z}(t) = Az(t) + Bv(t), \ z(s) = 0. \quad (6)$$

Proof: Let $x_u(t)$ denote the state trajectory of (5) in case the control $u(\cdot)$ is used. Then it is well-known that due to the linearity of the system

$$x_{\lambda u + (1-\lambda)w}(t) = \lambda x_u(t) + (1-\lambda)x_w(t). \quad (7)$$

From the definition of convexity (see (9)) it follows then that $J(0, T, u, x_0)$ is convex if and only if

$$\begin{aligned} & \int_0^T \{ [x_{\lambda u + (1-\lambda)w}^T(t) \ (\lambda u + (1-\lambda)w)^T(t)] M \begin{bmatrix} x_{\lambda u + (1-\lambda)w}(t) \\ (\lambda u + (1-\lambda)w)(t) \end{bmatrix} + \\ & \quad 2[p_1^T(t) \ p_2^T(t)] \begin{bmatrix} x_{\lambda u + (1-\lambda)w}(t) \\ (\lambda u + (1-\lambda)w)(t) \end{bmatrix} + c_2(t) \} dt \\ & \leq \int_0^T \{ \lambda [x_u^T(t) \ u^T(t)] M \begin{bmatrix} x_u(t) \\ u(t) \end{bmatrix} + 2[p_1^T(t) \ p_2^T(t)] \begin{bmatrix} x_u(t) \\ u(t) \end{bmatrix} + c_2(t) \} + \\ & \quad (1-\lambda) [x_w^T(t) \ w^T(t)] M \begin{bmatrix} x_w(t) \\ w(t) \end{bmatrix} + 2[p_1^T(t) \ p_2^T(t)] \begin{bmatrix} x_w(t) \\ w(t) \end{bmatrix} + c_2(t) \} dt. \end{aligned}$$

Or equivalently, using (7), we obtain after some rewriting

$$\lambda(1-\lambda) \int_0^T \{ [x_u^T(t) - x_w^T(t) \ u^T(t) - w^T(t)] M \begin{bmatrix} x_u(t) - x_w(t) \\ u(t) - w(t) \end{bmatrix} \} dt \geq 0,$$

where

$$\dot{x}_u(t) = Ax_u(t) + Bu(t) + c(t), \ x_u(0) = x_0 \text{ and } \dot{x}_w(t) = Ax_w(t) + Bw(t) + c(t), \ x_w(0) = x_0.$$

With $z := x_u - x_w$ and $v := u - w$ the stated result then follows immediately. \square

Remark 2.5

1. Note that the second part of the equivalence does not depend on x_0 . In particular it follows from this that if J is convex for one x_0 then J is convex for all x_0 . This property can also be verified by a direct elaboration of the convexity definition using the linearity property of the system again.

2. Consider a linear system in its state controllable canonical form

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t), \ x(0) = x_0, \text{ with } (A_1, B_1) \text{ controllable and} \\ J &= \int_0^T \{ x^T(t) \begin{bmatrix} Q_1 & Q_2 \\ Q_2 & Q_3 \end{bmatrix} x(t) + u^T(t) Ru(t) \} dt. \end{aligned}$$

Then it follows immediately from Lemma 2.4 that J is convex if and only if the controllable part of this system is convex. That is if, with

$$\begin{aligned}\dot{x}_1(t) &= A_1 x_1(t) + B_1 u(t), \quad x_1(0) = x_{10}, \text{ and} \\ J_1 &= \int_0^T \{x_1^T(t) Q_1 x_1(t) + u^T(t) R u(t)\} dt,\end{aligned}$$

J_1 is convex. □

Lemma 2.6 *Assume \mathcal{U} is convex. Consider the linear quadratic cost function (4) and (5). Then, if $J(0, T, u, x_0)$ is convex for some x_0 , $J(s, T, u, x_0)$ is convex for all $s \geq 0$ and for all x_0 .*

Proof: Let $s > 0$. Let $u_i(t) = v(t) + w_i(t)$, where $v(t) = 0$, $t \geq s$ and $w_i(t) = 0$, $0 \leq t \leq s$. Then,

$$J(0, T, u_i, x_0) = J(0, s, v, x_0) + J(s, T, w_i, x(s, x_0)). \quad (8)$$

Since $J(0, T, u, x_0)$ is convex, by definition, for an arbitrary u and w and $\lambda \in (0, 1)$

$$J(0, T, \lambda u + (1 - \lambda)w, x_0) \leq \lambda J(0, T, u, x_0) + (1 - \lambda)J(0, T, w, x_0). \quad (9)$$

So in particular if we choose $u = u_1$ and $w = u_2$ as above we get, using (8),

$$\begin{aligned}J(0, s, v, x_0) + J(s, T, \lambda w_1 + (1 - \lambda)w_2, x(s, x_0)) &= J(0, T, \lambda u_1 + (1 - \lambda)u_2, x_0) \\ &\leq \lambda J(0, T, u_1, x_0) + (1 - \lambda)J(0, T, u_2, x_0) \\ &= \lambda(J(0, s, v, x_0) + J(s, T, w_1, x(s, x_0))) + (1 - \lambda)(J(0, s, v, x_0) + J(s, T, w_2, x(s, x_0))) \\ &= J(0, s, v, x_0) + \lambda J(s, T, w_1, x(s, x_0)) + (1 - \lambda)J(s, T, w_2, x(s, x_0)).\end{aligned}$$

Comparing both sides of this inequality shows then that $J(s, T, w, x(s, x_0))$ is convex. The rest of the statement follows then directly from Remark 2.5, item 1. □

Corollary 2.7 *Consider the linear quadratic cost function (4) and (5). Then, $J(s, T, u, x_0)$ is convex for all $s \geq 0$ and x_0 if and only if $J(0, T, u, x_0)$ is convex for all/an x_0 , which holds if and only if $\bar{J}(0, T, v, 0) \geq 0$ for all v , where \bar{J} is given by (6). □*

Theorem 2.8 *Assume either T is finite or (A, B) is stabilizable. Then $J(0, T, u, x_0)$ is convex for all/an x_0 if and only if $\inf \bar{J}(0, T, v, 0)$ exists.*

Proof:

" \Rightarrow " From Corollary 2.7 it follows that if $J(0, T, u, x_0)$ is convex $\bar{J}(0, T, v, 0) \geq 0$ for all v . In case T is finite obviously $\inf \bar{J}(0, T, v, 0) \leq J(0, T, 0, 0)$ whereas in case (A, B) is stabilizable, $\inf \bar{J}(0, \infty, v, 0)$ is bounded from above too. Consequently, $\inf \bar{J}(0, T, v, 0)$ exists.

" \Leftarrow " Assume $\inf \bar{J}(0, T, v, 0) = m$. Then $m \geq 0$. For if $m < 0$ there would exist a \bar{v} such that $\bar{J}(0, T, \bar{v}, 0) < 0$. But from this it follows directly from the linearity of the system that $\bar{J}(0, T, \lambda \bar{v}, 0) = \lambda^2 \bar{J}(0, T, \bar{v}, 0)$. From which it is clear that $\inf \bar{J}(0, T, v, 0)$ would not exist. So $\bar{J}(0, T, v, 0) \geq 0$ for all v . This implies, see Corollary 2.7 again, that $J(0, T, u, x_0)$ is convex for all/an x_0 . □

Next we consider the case $R > 0$ in M . It is well-known that then the next Riccati equations play an important role

$$\dot{K}(t) = A^T K(t) + K(t)A - (K(t)B + V^T)R^{-1}(B^T K(t) + V) + Q, \quad K(T) = 0; \quad (\text{DRE})$$

$$0 = A^T K + KA - (KB + V^T)R^{-1}(B^T K + V) + Q. \quad (\text{ARE})$$

Let Γ denote the set of all symmetric solutions of (ARE). From, e.g., [1] and [10] (see also [8], [4], [12], [6] [7] and [11]) we recall the next results. Notice that these results are formulated in terms of existence of the performance function for an arbitrary initial state. In the above cited references one can also find conditions for the existence of $\inf \bar{J}(0, T, v, 0)$. Since the formulation of these conditions is more involved and is somewhat outside the main scope of this note they are not presented here.

Theorem 2.9 *Consider the linear quadratic cost function*

$$J := \int_0^T [x^T(t) \ u^T(t)] M \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt \text{ with } \dot{x}(t) = Ax(t) + Bu(t), \ x(0) = x_0. \quad (10)$$

Then the following holds:

- 1) *Let $T < \infty$ and $\mathcal{U} = L_2[0, T]$. Then, $\inf_u J$ exists for all x_0 if and only if (DRE) has a solution on $[0, T]$. Moreover, if this infimum exists it is in fact a minimum. This $\min_u J(u) = x_0^T K(0)x_0$ is attained uniquely by $u^*(t) = -R^{-1}(B^T K(t) + V)x^*(t)$, where $K(t)$ is the unique symmetric solution of (DRE) and $x^*(\cdot)$ solves $\dot{x}(t) = (A - BR^{-1}(B^T K(t) + V))x(t)$, $x^*(0) = x_0$.*
- 2) *Let $T = \infty$, $\mathcal{U} = L_{2,e,s}^+$ and (A, B) be stabilizable. Then, $\inf_u J$ exists for all x_0 if and only if $\Gamma \neq \emptyset$. Moreover, there exists a u^* attaining this infimum if and only if (ARE) has a stabilizing solution K^+ . Under this condition $\min_u J(u) = x_0^T K^+ x_0$ is attained uniquely by $u^*(t) = -R^{-1}(B^T K^+ + V)x^*(t)$, where $x^*(\cdot)$ solves $\dot{x}(t) = (A - BR^{-1}(B^T K^+ + V))x(t)$, $x^*(0) = x_0$.*
- 3) *Let $T = \infty$, $\mathcal{U} = L_{2,e}^+$ and (A, B) be controllable. Then, $\inf_u J$ exists for all x_0 if (ARE) has a symmetric solution $K \leq 0$. Moreover, if this condition is satisfied there exists a u^* attaining this infimum if and only if $\Delta \subset K^-$. Here K^- is the smallest solution of Γ and $\Delta := K^+ - K^-$, where K^+ is the largest solution of Γ . In that case $\min_u J(u) = x_0^T K_f x_0$ is attained uniquely by $u^*(t) = -R^{-1}(B^T K_f + V)x^*(t)$, where $x^*(\cdot)$ solves $\dot{x}(t) = (A - BR^{-1}(B^T K_f + V))x(t)$, $x^*(0) = x_0$ and $K_f = K^- P_N + K^+(I - P_N)$. Here P_N is the projector onto the subspace N along $\Delta^{-1}N^\perp$ with N the undetectable (w.r.t. $\mathbb{C}^- \cup \mathbb{C}^0$) subspace of $(K^-, (A - BR^{-1}(B^T K^- + V)))$. \square*

Remark 2.10 If (ARE) has a solution $K \geq 0$ then K_f in Theorem 2.9.3) is the smallest positive semi-definite solution of (ARE).

If one merely assumes (A, B) to be stabilizable instead of controllable in Theorem 2.9.3) (ARE) has in general not a smallest solution anymore. It can still be shown under the same condition as in 3) (that (ARE) has a solution $K \leq 0$) that the infimum exists and equals $x_0^T K_f x_0$ for some $K_f \in \Gamma$. However there does not exist a nice characterization like in 3) of K_f (see [3]). \square

Next introduce $B := [B_1 \ B_2]$. Combining the results of Lemmas 2.1, 2.2 and Theorems 2.8, 2.9 one can derive now straightforwardly existence and computational algorithms for both the finite and infinite horizon game problems. We will just state the result for the infinite planning horizon case where the state converges to zero. The formulation of the other results is left to the reader.

Corollary 2.11 Consider the cooperative game (1,2) with $T = \infty$, $\mathcal{U} = L_{2,e,s}^+$ and (A, B) stabilizable. Assume (ARE), with $V := [V_i \ W_i]$ and $R := \begin{bmatrix} R_{1i} & N_i \\ N_i^T & R_{2i} \end{bmatrix}$ has a stabilizing³ solution K_i , for $i = 1, 2$, respectively.

Then all Pareto efficient solutions are obtained by determining for $\alpha \in \mathcal{A}$

$$u^*(\alpha) := \arg \min_{u \in \mathcal{U}} \alpha_1 J_1 + \alpha_2 J_2, \text{ subject to (2).} \quad (11)$$

The corresponding Pareto solutions are

$$(J_1(u^*(\alpha)), J_2(u^*(\alpha))). \quad \square$$

Notice that, since by assumption both cost functions are strict convex, the minimization problem (11) has for all $\alpha \in \mathcal{A}$ a unique solution. From this result one obtains then the next procedure to calculate all Pareto efficient outcomes for this game.

Theorem 2.12 (Solution Cooperative Game)

Consider the cooperative game (1,2) with $T = \infty$, $\mathcal{U} = L_{2,e,s}^+$ and (A, B) stabilizable.

For $\alpha \in \mathcal{A}$ let

$$M(\alpha) := \alpha_1 M_1 + \alpha_2 M_2 =: \begin{bmatrix} \tilde{Q} & \tilde{V} \\ \tilde{V}^T & \tilde{R} \end{bmatrix},$$

where

$$\tilde{Q} = \alpha_1 Q_1 + \alpha_2 Q_2, \quad \tilde{V} = [\alpha_1 V_1 + \alpha_2 V_2, \quad \alpha_1 W_1 + \alpha_2 W_2], \quad \text{and} \quad \tilde{R} = \alpha_1 \begin{bmatrix} R_{11} & N_1 \\ N_1^T & R_{21} \end{bmatrix} + \alpha_2 \begin{bmatrix} R_{12} & N_2 \\ N_2^T & R_{22} \end{bmatrix}.$$

Furthermore, let $\tilde{S} := B\tilde{R}^{-1}B^T$.

Assume that (12) below has a stabilizing solution X_i for $\alpha_i = 1$, $i = 1, 2$, respectively.

$$A^T X + X A - (X B + \tilde{V}) \tilde{R}^{-1} (B^T X + \tilde{V}^T) + \tilde{Q} = 0. \quad (12)$$

Then the set of all cooperative Pareto solutions is given by

$$\{(J_1(u^*(\alpha)), J_2(u^*(\alpha))) \mid \alpha \in \mathcal{A}\}.$$

Here

$$u^*(t) = -\tilde{R}^{-1} (B^T X_s + \tilde{V}^T) x(t) - \tilde{R}^{-1} B^T \int_t^\infty e^{-A_{cl}^T(t-s)} X_s c(s) ds$$

where X_s is the stabilizing solution of (12) and, with $A_{cl} := A - B\tilde{R}^{-1}\tilde{V}^T - \tilde{S}X_s$, the closed-loop system is $\dot{x}(t) = A_{cl}x(t) - Sm(t) + c(t)$, $x(0) = x_0$. In case $c(\cdot) = 0$ the with these actions corresponding cost are $J_i(x_0, u^*) = x_0^T \tilde{M}_i x_0$, where \tilde{M}_i is the unique solution of the Lyapunov equation

$$A_{cl}^T \tilde{M}_i + \tilde{M}_i A_{cl} = -[I, -(X_s B + \tilde{V}) \tilde{R}^{-1}] M_i \begin{bmatrix} I \\ -\tilde{R}^{-1} (B^T X_s + \tilde{V}^T) \end{bmatrix}. \quad \square$$

³That is, $\sigma(A - BR^{-1}V^T - SK_i) \subset \mathbb{C}^-$.

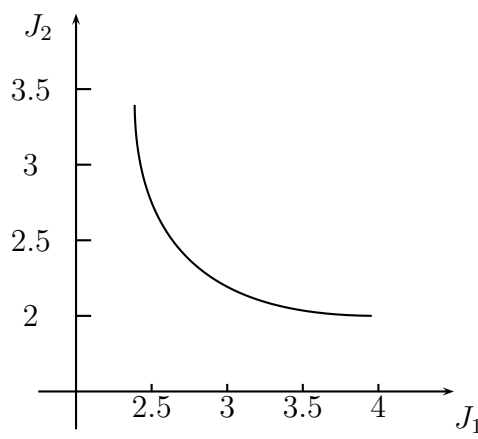


Figure 1: Pareto frontier Example 2.13.

Example 2.13 To illustrate the above result consider the next simplistic environmental example. Consider a fishery management game, where two industries pollute a lake and their income depends on the fish-stock of the lake. Assume that both the revenues and cost depend quadratically on the fish-stock and the produced pollution. Within this context we consider the next model:

$$\dot{x}(t) = 2x(t) - u_1(t) - u_2(t), \quad x(0) = 1,$$

with cost functions

$$J_1 = \int_0^\infty -x^2(t) + 2u_1^2(t) + u_2^2(t)dt \quad \text{and} \quad J_2 = \int_0^\infty -2x^2(t) + u_1^2(t) + 2u_2^2(t)dt.$$

Here x describes the fish-stock and u_i is the pollution produced by industry i . The fact that industry 1 gives more weight to the pollution it produces than to the quantity produced by the other industry can be interpreted as that industry 1 is really concerned about the pollution it produces itself (and which it is able to control in contrast to the pollution produced by the other industry). In figure 1 we plotted the with this problem corresponding Pareto frontier if both players would cooperate in this game. \square

3 Concluding Remarks

In this note we showed how for the regular indefinite linear quadratic control problem one can obtain all Pareto efficient outcomes. Using the theory of convex analysis we were able to treat both the finite and infinite planning horizon problems in a uniform way. For the infinite planning horizon we considered both the fixed endpoint problem and the free endpoint problem.

Open problems that remain to be solved in this context are, e.g., the free endpoint infinite planning horizon problem under the assumption that the system is merely stabilizable and the case that the performance criteria of both players are not directly affected by the control efforts used by the other player (giving rise to a special singular control problem). To solve the firstmentioned problem a good starting point might be to find first a different characterization of the solution of the free endpoint problem under the controllability assumption. Solving the lastmentioned problem is a point of current research. We hope to solve this problem using a different analytic approach. A number of necessary conditions for Pareto optima in a general dynamic framework have been derived and current research focusses on the question whether these conditions are sufficient too for that specific linear quadratic framework.

References

- [1] Engwerda J.C., 2005, *LQ Dynamic Optimization and Differential Games*, Chichester, Wiley.
- [2] Fan K., Glicksberg I. and Hoffman A.J., 1957, Systems of inequalities involving convex functions, *American Mathematical Society Proceedings*, Vol.8.
- [3] Geerts T., 1991, A priori results in linear-quadratic optimal control theory, *Kybernetika*, vol.27, pp.446-457.
- [4] Jacobson D.H., 1971, A general sufficiency theorem for the second variation, *J. Math. Anal. Appl.*, vol.34, pp.578-589.
- [5] Leitmann G., 1974, *Cooperative and Non-cooperative Many Players Differential Games*, Springer Verlag, Berlin.
- [6] Molinari B.P., 1975, Nonnegativity of a quadratic functional, *SIAM J. Control and Optimization*, vol.13, pp.792-806.
- [7] Molinari B.P., 1977, The time-invariant linear-quadratic optimal control problem, *Automatica*, vol.13, pp.347-357.
- [8] Pars L.A., 1962, *An Introduction to the Calculus of Variations*, Heinemann, London.
- [9] Takayama A., 1985, *Mathematical Economics*, Cambridge University Press, Cambridge.
- [10] Trentelman H.L., 1989, The regular free-endpoint linear quadratic problem with indefinite cost, *SIAM J. Control and Optimization*, vol.27, pp.27-42.
- [11] Trentelman H.L. and Willems J.C., 1991, The dissipation inequality and the algebraic Riccati equation. In S. Bittanti, A.J. Laub and J.C. Willems (Eds.), *The Riccati Equation*, Springer Verlag, Berlin, pp.197-242.
- [12] Willems J.C., 1971, Least squares stationary optimal control and the algebraic Riccati equation, *IEEE Transactions on Automatic Control*, vol.16, pp.641-634.
- [13] Yeung D.W.K and Petrosyan L.A., 2005, Subgame consistent solutions of a cooperative stochastic differential game with nontransferable payoffs, *Journal of Optimization Theory and Applications*, vol.124, pp.701-724.